

A NOTE ON LIOUVILLE TYPE THEOREM OF ELLIPTIC INEQUALITY $\Delta u + u^\sigma \leq 0$ ON RIEMANNIAN MANIFOLDS

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ABSTRACT. Let $\sigma > 1$ and let M be a complete Riemannian manifold. In a very recent work [10], Grigor'yan and Sun proved that a Liouville type theorem for nonnegative solutions of elliptic inequality

$$(*) \quad \Delta u(x) + u^\sigma(x) \leq 0, \quad x \in M.$$

via a pointwise condition of volume growth of geodesic balls. In this note, we improve their result to that an *integral condition* on volume growth implies the same uniqueness of (*). It is inspired by the well-known Varopoulos-Grigor'yan's criterion for parabolicity of M .

1. INTRODUCTION

Let $\sigma > 1$ and let M be a complete noncompact Riemannian manifold without boundary. Consider the semilinear elliptic inequality

$$(1.1) \quad \Delta u(x) + u^\sigma(x) \leq 0, \quad x \in M,$$

where Δ is the Laplace-Beltrami operator on M . A function $u \in W_{\text{loc}}^{1,2}(M)$ is called a *weak solution* of the inequality (1.1) if

$$-\int_M \langle \nabla u, \nabla \psi \rangle d\mu + \int_M u^\sigma \psi d\mu \leq 0$$

holds for any nonnegative function $\psi \in W^{1,2}(M)$ with compact support.

In Euclidean setting, i.e. $M = \mathbb{R}^n$, it has a long history to study the uniqueness of nonnegative solutions for (1.1) (or more general elliptic inequalities and equalities). There are many beautiful results have been obtained in this subject. We refer the readers to, for instance, [1, 2, 3, 4, 11, 12, 13] and references therein for them. Many of these results are based on comparison principle and careful choices of test functions for (1.1). To use this method on a manifold M , one have to estimate the second order derivative of distance functions, which needs some assumptions on curvature of M .

Surprisingly, in recent works Grigor'yan-Kondratiev [9] and Grigor'yan-Sun [10] proved a *curvature-free* Liouville type theorem for nonnegative weak solution of (1.1) in terms of volume growth of geodesic balls in M as follows.

Theorem 1.1 (Grigor'yan-Sun [10]). *Let M be a complete Riemannian manifold without boundary. Fix a point $x_0 \in M$ and set $V(r) := \mu(B(x_0, r))$ the volume of geodesic ball of radius r centered at x_0 .*

Assume that, for some $C > 0$, the inequality

$$(1.2) \quad V(r) \leq Cr^{\frac{2\sigma}{\sigma-1}} (\ln r)^{\frac{1}{\sigma-1}}$$

holds for all large enough r . Then any nonnegative weak solution of (1.1) is identically equal to 0.

They also showed that the exponents $\frac{2\sigma}{\sigma-1}$ and $\frac{1}{\sigma-1}$ are sharp.

On the other hand, let us recall that a manifold M is said to be *parabolic* if a Liouville type theorem holds for nonnegative solution of inequality

$$\Delta u(x) \leq 0, \quad x \in M,$$

i.e., any nonnegative weak solution of $\Delta u \leq 0$ on M must be constant. Cheng and Yau [5] proved that $V(r) \leq Cr^2$, for some $C > 0$, is a sufficient condition for parabolicity of M . Nowadays, a well-known sharp sufficient condition for parabolicity is the following *integral* condition, which was proved independly by Varopoulos [14] and Grigor'yan [7, 8]:

$$\int_1^\infty \frac{r}{V(r)} dr = \infty.$$

Inspired by Varopoulos-Grigor'yan's condition for the parabolicity of M , we ask a natural question: what is a sufficient condition for Liouville type theorem of inequality (1.1) via an *integral* estimate of $V(r)$? Of course, such a condition should cover the above pointwise condition (1.2).

In this remark, we solve this question. Our main result states as follows:

Theorem 1.2. *Let M be a complete Riemannian manifold without boundary. Assume that*

$$(1.3) \quad \liminf_{t \rightarrow 0^+} t^{\frac{\sigma}{\sigma-1}} \int_1^\infty \frac{V(r)}{r^{\frac{3\sigma-1}{\sigma-1}+t}} dr < \infty.$$

Then any nonnegative weak solution of (1.1) is identically equal to 0.

Remark 1.3. Condition (1.2) in Theorem 1.1 implies the condition (1.3). In fact,

$$(1.2) \implies t^{\frac{\sigma}{\sigma-1}} \int_1^\infty \frac{V(r)dr}{r^{\frac{3\sigma-1}{\sigma-1}+t}} \leq t^{\frac{\sigma}{\sigma-1}} \int_1^\infty \frac{(\ln r)^{\frac{1}{\sigma-1}} dr}{r^{1+t}} = \Gamma\left(\frac{\sigma}{\sigma-1}\right),$$

where $\Gamma(\cdot)$ is Gamma function.

2. PROOF OF THEOREM 1.2

Proof of Theorem 1.2. Let $u \in W_{\text{loc}}^{1,2}(M)$ be a nontrivial nonnegative solution to the inequality (1.1).

The proof of Theorem 1.1 in [10] contains two main parts. Firstly, the authors derived a useful priori estimate in terms of a test function and positive parameters (which will be recalled in Lemma 2.1 below). Secondly, they chose specific test functions to conclude $\int_M u^\sigma d\mu = 0$. Our proof of Theorem 1.2 is basically along the same line in [10]. The different from

Grigor'yan-Sin's proof will appear in the second part. We will choose a variation of their test functions to conclude $\int_M u^\sigma d\mu = 0$.

Firstly, let us recall the useful priori estimate given in [10]. We summarize it as the following lemma:

Lemma 2.1 (Grigor'yan-Sun, [10]). *Set $s = 8\sigma/(\sigma - 1)$. Then there exists a constant $C_0 > 0$ such that the following property holds:*

For any

$$t \in (0, \min\{1, \frac{\sigma - 1}{2}\}),$$

any nonempty compact set $K \subset M$, and any Lipschitz function ϕ on M with compact support such that $0 \leq \phi \leq 1$ on M and $\phi \equiv 1$ in a neighborhood of K , we have

$$(2.1) \quad \int_M \phi^s u^\sigma d\mu \leq C_0 \left(\int_{M \setminus K} \phi^s u^\sigma d\mu \right)^{\frac{t+1}{2\sigma}} \cdot J(t, \phi)$$

and

$$(2.2) \quad \left(\int_M \phi^s u^\sigma d\mu \right)^{1 - \frac{t+1}{2\sigma}} \leq C_0 \cdot J(t, \phi),$$

where

$$J(t, \phi) := t^{-\frac{1}{2} - \frac{\sigma}{2(\sigma-1)}} \left(\int_M |\nabla \phi|^2 \frac{\sigma-t}{\sigma-1} d\mu \right)^{\frac{1}{2}} \cdot \left(\int_M |\nabla \phi|^{\frac{2\sigma}{\sigma-t-1}} d\mu \right)^{\frac{\sigma-t-1}{2\sigma}}.$$

Proof. Inequality (2.1) is Eq.(2.10) in [10], and inequality (2.2) is Eq.(2.11) in [10]. \square

In the following, we will consider a family of specific test functions ϕ_n , which are modifications from original structures in [10].

Fix any $t \in (0, \min\{1, \frac{\sigma-1}{2}\})$. We set $R = R(t) := \exp(1/t)$. We consider the function

$$\phi_t(x) = \begin{cases} 1, & r(x) < R, \\ \left(\frac{r(x)}{R}\right)^{-t}, & r(x) \geq R, \end{cases}$$

and a family of functions, for any $n = 1, 2, 3, \dots$,

$$\xi_{t,n}(x) = \begin{cases} 1, & 0 \leq r(x) \leq 2^n R, \\ 2 - \frac{r(x)}{2^n R}, & 2^n R \leq r(x) \leq 2^{n+1} R, \\ 0, & r(x) \geq 2^{n+1} R. \end{cases}$$

Consider the functions

$$(2.3) \quad \phi_{t,n}(x) := \phi_t(x) \cdot \xi_{t,n}(x).$$

Then, for each $n = 1, 2, \dots$, function $\phi_{t,n}(x)$ is Lipschitz continuous on M and has compact support, and $\phi_{t,n} \equiv 1$ on $B_{R(t)} := B(x_0, R(t))$.

Claim: *There exists a constant $C_1 > 0$ such that, for any $t \in (0, \min\{1, \frac{\sigma-1}{2}\})$ with*

$$A(t) := \int_1^\infty \frac{V(r)}{r^{\frac{3\sigma-1}{\sigma-1}+t}} dr < \infty,$$

we have

$$(2.4) \quad \limsup_{n \rightarrow \infty} [J(t, \phi_{t,n})]^{\frac{2\sigma}{2\sigma-t-1}} \leq C_1 \cdot t^{\frac{\sigma}{\sigma-1}} \cdot A(t).$$

Proof of Claim: In the proof, the parameter t is fixed. To simplify the notations, we denote by

$$\phi := \phi_t, \quad \xi_n := \xi_{t,n} \quad \text{and} \quad \phi_n := \phi_{t,n}.$$

Notice that

$$\nabla \phi_n = \xi_n \cdot \nabla \phi + \phi \cdot \nabla \xi_n.$$

We have

$$|\nabla \phi_n| \leq \xi_n \cdot |\nabla \phi| + \phi \cdot |\nabla \xi_n|;$$

and, by the inequality $(A+B)^a \leq 2^{a-1}(A^a+B^a)$ for all $A, B > 0$ and $a \geq 1$,

$$|\nabla \phi_n|^a \leq 2^{\frac{4\sigma}{\sigma-1}-1} [\xi_n^a \cdot |\nabla \phi|^a + \phi^a \cdot |\nabla \xi_n|^a]$$

for any $a \in [1, \frac{4\sigma}{\sigma-1}]$. In the following, we denote by

$$\sigma_0 := \frac{4\sigma}{\sigma-1}.$$

Similar as in [10], we need to estimate the integral $\int_M |\nabla \phi_n|^a d\mu$. For any $a \in [1, \sigma_0]$, we have

$$(2.5) \quad \begin{aligned} \int_M |\nabla \phi_n|^a d\mu &\leq 2^{\sigma_0-1} \cdot \left(\int_{M \setminus B_R} |\nabla \phi|^a d\mu + \int_{B_{2^{n+1}R} \setminus B_{2^n R}} \phi^a |\nabla \xi_n|^a d\mu \right) \\ &:= 2^{\sigma_0-1} \cdot (I(a) + II(a, n)), \end{aligned}$$

where $B_R := B(x_0, R)$, and we have used that $\nabla \phi = 0$ in B_R and that $|\nabla \xi_n|$ supported in $\overline{B_{2^{n+1}R}} \setminus B_{2^n R}$.

Before we estimate the above integrals $I(a)$ and $II(a, n)$, we need the following simple (but important) observation:

If the parameter $a \in [1, \sigma_0]$ satisfies

$$(2.6) \quad a(t+1) \geq t + \frac{2\sigma}{\sigma-1}.$$

Then we have

$$(2.7) \quad \sum_{n=1}^{\infty} \frac{V(2^n R)}{(2^{n-1} R)^{a(t+1)}} \leq 2 \cdot 16^{\sigma_0} \cdot A(t) := C_2 \cdot A(t).$$

In particular, it implies that

$$(2.8) \quad \lim_{n \rightarrow \infty} \frac{V(2^n R)}{(2^{n-1} R)^{a(t+1)}} = 0.$$

Indeed, we calculate directly to conclude

$$\begin{aligned}
 (2.9) \quad & \sum_{n=1}^{\infty} \frac{V(2^n R)}{(2^{n-1} R)^{a(t+1)}} \\
 &= 4^{a(t+1)} \cdot 2 \cdot \sum_{n=1}^{\infty} \frac{V(2^n R)}{(2^{n+1} R)^{a(t+1)}} \cdot \frac{2^{n+1} R - 2^n R}{2^{n+1} R} \\
 &\leq 4^{a(t+1)} \cdot 2 \cdot \sum_{n=1}^{\infty} \int_{2^n R}^{2^{n+1} R} \frac{V(r) dr}{r^{a(t+1)+1}} \\
 &\leq 2 \cdot 16^{\sigma_0} \cdot \int_1^{\infty} \frac{V(r) dr}{r^{a(t+1)+1}},
 \end{aligned}$$

we have used that $t < 1$, $a \leq \sigma_0$ and that $R = \exp(1/t) > 1$. Combining with (2.6) and (2.9), we can obtain

$$\sum_{n=1}^{\infty} \frac{V(2^n R)}{(2^{n-1} R)^{a(t+1)}} \leq 2 \cdot 16^{\sigma_0} \int_1^{\infty} \frac{V(r) dr}{r^{t+\frac{2\sigma}{\sigma-1}+1}} = 2 \cdot 16^{\sigma_0} \cdot A(t).$$

This is the desired estimate (2.7).

Now let us estimate $I(a)$. Assume that the parameter a satisfies (2.6), we have

$$\begin{aligned}
 (2.10) \quad I(a) &= \int_{M \setminus B_R} |\nabla \phi|^a d\mu \leq \int_{M \setminus B_R} \left[\frac{t}{R} \cdot \left(\frac{r}{R} \right)^{-t-1} \right]^a d\mu \\
 &= e^a \cdot t^a \int_{M \setminus B_R} \frac{1}{r^{a(t+1)}} d\mu \quad (\text{since } R^t = e) \\
 &= e^a \cdot t^a \cdot \sum_{n=1}^{\infty} \int_{B_{2^n R} \setminus B_{2^{n-1} R}} \frac{1}{r^{a(t+1)}} d\mu \\
 &\leq e^a \cdot t^a \cdot \sum_{n=1}^{\infty} \frac{V(2^n R)}{(2^{n-1} R)^{a(t+1)}} \\
 &\leq e^{\sigma_0} \cdot C_2 \cdot t^a A(t) \quad (\text{by } a \leq \sigma_0 \text{ and (2.7)}).
 \end{aligned}$$

Let us estimate $II(a, n)$. Assume that the parameter a satisfies (2.6), we have

$$\begin{aligned}
 (2.11) \quad II(a, n) &= \int_{B_{2^{n+1} R} \setminus B_{2^n R}} \phi^a |\nabla \xi_n|^a d\mu \\
 &\leq \left(\frac{2^n R}{R} \right)^{-at} \left(\frac{1}{2^n R} \right)^a \cdot V(2^{n+1} R) \\
 &= R^{at} \cdot \frac{V(2^{n+1} R)}{(2^n R)^{a(t+1)}} \stackrel{R^t=e}{=} e^a \cdot \frac{V(2^{n+1} R)}{(2^n R)^{a(t+1)}}.
 \end{aligned}$$

Combining with (2.8), (2.11) and that $a \leq \sigma_0$, we have

$$(2.12) \quad \lim_{n \rightarrow \infty} II(a, n) = 0.$$

Therefore, according to (2.5), (2.10) and (2.12), we obtain, for any $a \in [1, \sigma_0]$ satisfying (2.6),

$$(2.13) \quad \limsup_{n \rightarrow \infty} \int_M |\nabla \phi_n|^a d\mu \leq 2^{\sigma_0-1} \cdot e^{\sigma_0} \cdot C_2 \cdot t^a A(t) := C_3 \cdot t^a A(t).$$

We take

$$a_1 = 2 \frac{\sigma - t}{\sigma - 1} \quad \text{and} \quad a_2 = \frac{2\sigma}{\sigma - t - 1}.$$

Then it is easy to check that a_1, a_2 satisfy (2.6). Indeed,

$$a_1(t+1) = \frac{2\sigma}{\sigma-1} + 2t - \frac{2t^2}{\sigma-1} \geq \frac{2\sigma}{\sigma-1} + t \quad (\text{since } t \leq \frac{\sigma-1}{2})$$

and

$$a_2(t+1) = \frac{2\sigma}{\sigma-1} \cdot \frac{\sigma-1}{\sigma-t-1} \cdot (t+1) \geq \frac{2\sigma}{\sigma-1} \cdot (t+1) \geq \frac{2\sigma}{\sigma-1} + t.$$

Now, by using

$$J(t, \phi_n) = t^{-\frac{1}{2} - \frac{\sigma}{2(\sigma-1)}} \left(\int_M |\nabla \phi_n|^{a_1} d\mu \right)^{\frac{1}{2}} \cdot \left(\int_M |\nabla \phi_n|^{a_2} d\mu \right)^{\frac{1}{a_2}}$$

and (2.13), we can conclude that

$$\begin{aligned} \limsup_{n \rightarrow \infty} J(t, \phi_n) &\leq t^{-\frac{1}{2} - \frac{\sigma}{2(\sigma-1)}} \cdot C_3^{\frac{1}{2} + \frac{1}{a_2}} \cdot t^{\frac{a_1}{2} + 1} \cdot [A(t)]^{\frac{1}{2} + \frac{1}{a_2}} \\ &= C_3^{\frac{2\sigma-t-1}{2\sigma}} \cdot t^{\frac{1}{2} + \frac{\sigma}{2(\sigma-1)} - \frac{t}{\sigma-1}} \cdot [A(t)]^{\frac{2\sigma-t-1}{2\sigma}} \end{aligned}$$

Then

$$\begin{aligned} (2.14) \quad \limsup_{n \rightarrow \infty} [J(t, \phi_n)]^{\frac{2\sigma}{2\sigma-t-1}} &\leq C_3 \cdot t^{(\frac{1}{2} + \frac{\sigma}{2(\sigma-1)} - \frac{t}{\sigma-1}) \cdot \frac{2\sigma}{2\sigma-t-1}} \cdot A(t) \\ &= C_3 \cdot t^{\frac{\sigma}{\sigma-1} \cdot (1 - \frac{t}{2\sigma-t-1})} \cdot A(t). \end{aligned}$$

Noticing that

$$\lim_{t \rightarrow 0^+} t^{-\frac{\sigma}{\sigma-1} \cdot \frac{t}{2\sigma-t-1}} = 1,$$

we have that the function $t \mapsto t^{-\frac{\sigma}{\sigma-1} \cdot \frac{t}{2\sigma-t-1}}$ is bounded on $(0, 1)$ uniformly.

Set the constant

$$C_1 := C_3 \cdot \sup_{0 < t < 1} t^{-\frac{\sigma}{\sigma-1} \cdot \frac{t}{2\sigma-t-1}}.$$

Then the desired estimate (2.4) follows from (2.14), and hence the proof of **Claim** is completed. \square

Now let us continue the proof of Theorem 1.2.

According to (1.3), there is a sequence of numbers $\{t_\alpha\}_{\alpha=1}^\infty$, going to 0, such that

$$(2.15) \quad t_\alpha^{\frac{\sigma}{\sigma-1}} \cdot A(t_\alpha) = t_\alpha^{\frac{\sigma}{\sigma-1}} \int_1^\infty \frac{V(r)}{r^{\frac{3\sigma-1}{\sigma-1} + t_\alpha}} dr \leq C_4, \quad \forall \alpha = 1, 2, \dots$$

for some constant C_4 , independent of α . Without loss the generality, we can also assume that $t_\alpha \in (0, \min\{1, \frac{\sigma-1}{2}\})$, for all $\alpha = 1, 2, 3, \dots$.

By using the above **Claim**, we have, for each $\alpha = 1, 2, \dots$,

$$(2.16) \quad \limsup_{n \rightarrow \infty} J(t_\alpha, \phi_{t_\alpha, n}) \leq (C_1 \cdot C_4)^{\frac{2\sigma - t_\alpha - 1}{2\sigma}} \leq \max\{(C_1 C_4)^{\frac{2\sigma - 1}{2\sigma}}, 1\} := C_5.$$

In the following is similar as in [10]. We want to show $u \in L^\sigma(M)$, and moreover $\int_M u^\sigma d\mu = 0$. Fix arbitrary a nonempty compact set $K \subset M$.

Notice that $R(t_\alpha) = \exp(1/t_\alpha) \rightarrow \infty$ as $\alpha \rightarrow \infty$. So, we have

$$K \subset B_{R(t_\alpha)}$$

for all large enough α . Hence, for any sufficient large α , $\phi_{t_\alpha, n} \equiv 1$ on K holds for any $n = 1, 2, \dots$. For such α , we can apply Lemma 2.1 to t_α , K and function $\phi_{t_\alpha, n}$; and we conclude that

$$(2.17) \quad \int_K u^\sigma d\mu \leq \int_M \phi_{t_\alpha, n}^s u^\sigma d\mu \leq C_0 \left(\int_{M \setminus K} \phi_{t_\alpha, n}^s u^\sigma d\mu \right)^{\frac{t_\alpha + 1}{2\sigma}} \cdot J(t_\alpha, \phi_{t_\alpha, n})$$

and

$$(2.18) \quad \int_K u^\sigma d\mu \leq \int_M \phi_{t_\alpha, n}^s u^\sigma d\mu \leq \left(C_0 \cdot J(t_\alpha, \phi_{t_\alpha, n}) \right)^{\frac{2\sigma}{2\sigma - t_\alpha - 1}},$$

for all $n = 1, 2, \dots$, where we have used that $\phi_{t_\alpha, n} \equiv 1$ on K .

By combining (2.16) and (2.18), we obtain

$$\int_K u^\sigma d\mu \leq \left(C_0 \cdot C_5 \right)^{\frac{2\sigma}{2\sigma - t_\alpha - 1}}$$

for all large enough α . Letting $\alpha \rightarrow \infty$, we have

$$(2.19) \quad \int_K u^\sigma d\mu \leq \left(C_0 \cdot C_5 \right)^{\frac{2\sigma}{2\sigma - 1}} := C_6.$$

By combining with (2.17), (2.19), (2.16) and that $\phi_{t_\alpha, n} \leq 1$ on M , we have

$$\int_K u^\sigma d\mu \leq C_0 \left(\int_{M \setminus K} u^\sigma d\mu \right)^{\frac{t_\alpha + 1}{2\sigma}} \cdot C_5,$$

for all large enough α . Letting $\alpha \rightarrow \infty$, we have

$$(2.20) \quad \int_K u^\sigma d\mu \leq C_0 \cdot C_5 \cdot \left(\int_{M \setminus K} u^\sigma d\mu \right)^{\frac{1}{2\sigma}}.$$

By using the arbitrariness of K , we can take $K = \overline{B_r}$ for any $r > 0$. Combining with (2.19) and (2.20) and letting $r \rightarrow \infty$, we have

$$\int_M u^\sigma d\mu = 0,$$

which implies $u \equiv 0$ on M , and the proof of Theorem 1.2 is completed. \square

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REFERENCES

- [1] S. Alarcón; J. García-Melián & A. Quaas, *Nonexistence of positive supersolutions to some nonlinear elliptic problems*, J. Math. Pures Appl. 99(2013) 618–634.
- [2] Bidaut-Veron, M.-F., & Pohozaev, S., *Nonexistence results and estimates for some nonlinear elliptic problems*, J. Anal. Math., 84 (2001), 1–49.
- [3] Caffarelli, L.; Garofalo, N. & Segala, F., *A gradient bound for entire solutions of quasilinear equations and its consequences*, Comm. Pure Appl. Math., 47 (1994), 1457–1473.
- [4] Caristi, G.; Mitidieri, E. & Pohozaev, S. I., *Some Liouville theorems for quasilinear elliptic inequalities*, Doklady Math. 79 (2009), no. 1, 118–124.
- [5] S. Y. Cheng & S. T. Yau, *Differential equations on Riemannian manifolds and their geometric applications*, Comm. Pure Appl. Math., 28(1975), no. 3, 333–354.
- [6] Gidas, B. & Spruck, J., *Global and local behavior of positive solutions of nonlinear elliptic equations*, Comm. Pure Appl. Math. 34 (1981), no. 4, 525–598.
- [7] A. Grigor'yan, *On the existence of a Green function on a manifold* (in Russian), Uspekhi Math. Nauk 38(1)(1983) 161–162, Engl. transl.: Russian Math. Surveys 38(1)(1983) 190–191.
- [8] A. Grigor'yan, *On the existence of positive fundamental solution of the Laplace equation on Riemannian manifolds* (in Russian), Mat. Sb. 128(3)(1985) 354–363, Engl. transl.: Math. USSR Sb. 56 (1987) 349–358.
- [9] A. Grigor'yan & V. A. Kondratiev, *On the existence of positive solutions of semilinear elliptic inequalities on Riemannian manifolds*. Around the research of Vladimir Mazya II, 203–218. International Mathematical Series (New York), 12. Springer, New York, 2010.
- [10] A. Grigor'yan & Y. Sun, *On Nonnegative Solutions of the Inequality $\Delta u + u^\sigma \leq 0$ on Riemannian Manifolds*, to appear in Comm. Pure Appl. Math., (2013).
- [11] Pohozaev, S. I., *Critical nonlinearities in partial differential equations*, Milan J. Math. 77 (2009), 127–150.
- [12] J. Serrin, *Entire solutions of nonlinear Poisson equations*, Proc. London Math. Soc. (3), 24 (1972), 348–366.
- [13] J. Serrin & H. Zou, *Cauchy-Liouville and universal boundedness theorems for quasilinear elliptic equations and inequalities*, Acta Math., 189(2002), 79–142.
- [14] N. Varopoulos, *The Poisson kernel on positively curved manifolds*, J. Funct. Anal. 44(1981), 359–380.

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